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Prelude: What Is Algebra?

What is algebra? It is a question to which a high school student will give one answer, a college student majoring in mathematics another, and a professor who teaches graduate courses and conducts algebraic research a third. The educated "layperson," on the other hand, might simply grimace while retorting, "Oh, I never did well in mathematics. Wasn't algebra all of that *x* and *y* stuff that I could never figure out?" This ostensibly simple question, then, apparently has a number of possible answers. What do the "experts" say?

On 18 April 2006, the National Mathematics Advisory Panel (NMAP) within the US Department of Education was established by executive order of then President George W. Bush to advise him, as well as the Secretary of Education, on means to "foster greater knowledge of and improved performance in mathematics among American students."[1] Among the panel's charges was to make recommendations on "the critical skills and skill progressions for students to acquire competence in algebra and readiness for higher levels of mathematics." Why should competence in algebra have been especially singled out?

When it issued its final report in March 2008, the panel stated that "a strong grounding in high school mathematics through Algebra II or higher correlates powerfully with access to college, graduation from college, and earning in the top quartile of income from employment."[2] Furthermore, it acknowledged that "although our students encounter difficulties with many aspects of mathematics, many observers of

[1] US Dept. of Education, 2008, p. 71. The next quotation is also found here.

[2] US Dept. of Education, 2008, p. xii. For the next two quotations, see pp. xiii and 16, respectively.

educational policy see Algebra as a central concern." The panel had thus sought to determine how best to prepare students for entry into algebra and, since algebra was of such concern, it had first to come to terms with the question, what is the essential content of school algebra? In answer, it identified the following as the major topics: symbols and expressions, linear equations, quadratic equations, functions, the algebra of polynomials, and combinatorics and finite probability. Of course, each of these topics encompasses several subtopics. For example, the "algebra of polynomials" includes complex numbers and operations, the fundamental theorem of algebra, and Pascal's triangle. Interestingly, the panel mentioned "logarithmic functions" and "trigonometric functions" under the topic of "functions" but made no explicit mention of analytic geometry except in the special case of graphs of quadratic functions. Although the details of the panel's list might prompt these and other quibbles, it nevertheless gives some idea of what high school students, in the United States at least, generally study—or should study—under the rubric of "algebra."

These topics, however, constitute "school algebra." What about algebra at the college level? Most courses entitled "college algebra" in the United States simply revisit the aforementioned topics, sometimes going into slightly greater depth than is expected in high school. Courses for mathematics majors—entitled "modern algebra" or "abstract algebra"—are quite another matter, however. They embrace totally different topics: groups, rings, fields, and, often, Galois theory. Sometimes such courses also include vectors, matrices, determinants, and algebras (where the latter is a technical term quite different from the broad subject under consideration here).

And then there is algebra at the graduate and research levels. Graduate students may take courses in commutative or noncommutative algebra, representation theory, or Lie theory, while research mathematicians styled "algebraists" may deal with topics like "homological functors on modules," "algebraic coding theory," "regular local rings," or any one of hundreds of topics listed in the American Mathematical Society's "Mathematics Subject Classification." How do all of these subjects at all of these levels of sophistication fit together to constitute something called "algebra"? Before addressing this question, we might first ask why we need *this* book about it?

WHY THIS BOOK?

To be sure, the historical literature already includes several more or less widely ranging books on the history of algebra that are targeted, like the present book, at those with a background equivalent to a college major in mathematics;[3] a recent "popular" book assumes even less in the way of mathematical prerequisites.[4] Most in the former group, however, are limited either in the eras covered or in geographical reach, while that in the latter has too many errors of fact and interpretation to stand unchallenged. *This* book thus grew out of a shared realization that the time was ripe for a history of algebra that told the broader story by incorporating new scholarship on the diverse regions within which algebraic thought developed and by tracing the major themes into the early twentieth century with the advent of the so-called "modern algebra."

We also believe that this is a story very much worth telling, since it is a history very much worth knowing. Using the history of algebra, teachers of the subject, either at the school or at the college level, can increase students' overall understanding of the material. The "logical" development so prevalent in our textbooks is often sterile because it explains neither why people were interested in a particular algebraic topic in the first place nor why our students should be interested in that topic today. History, on the other hand, often demonstrates the reasons for both. With an understanding of the historical development of algebra, moreover, teachers can better impart to their students an appreciation that algebra is not arbitrary, that it is not created "full-blown" by fiat. Rather, it develops at the hands of people who need to solve vital problems, problems the solutions of which merit understanding. Algebra has been and is being created in many areas of the world, with the same solution often appearing in disparate times and places.

And this is neither a story nor a history limited to school students and their teachers. College-level mathematics students and their

[3] In fact, the prerequisites for reading the first ten chapters are little more than a solid high school mathematics education. The more general histories of algebra include van der Waerden, 1985; Scholz, 1990; Bashmakova and Smirnova, 2000; and Cooke, 2008, while the more targeted include Nový, 1973; Sesiano, 1999 and 2009; Kleiner, 2007; and Stedall, 2011.

[4] Derbyshire, 2006.

professors should also know the roots of the algebra they study. With an understanding of the historical development of the field, professors can stimulate their students to master often complex notions by motivating the material through the historical questions that prompted its development. In absorbing the idea, moreover, that people struggled with many important mathematical ideas before finding their solutions, that they frequently could not solve problems entirely, and that they consciously left them for their successors to explore, students can better appreciate the mathematical endeavor and its shared purpose. To paraphrase the great seventeenth- and early eighteenth-century English mathematician and natural philosopher, Sir Isaac Newton, mathematicians have always seen farther by "standing on the shoulders" of those who came before them.

One of our goals in the present book is thus to show how—in often convoluted historical twists and turns—the deeper and deeper consideration of some of the earliest algebraic topics—those generally covered in schools—ultimately led mathematicians to discover or invent the ideas that constitute much of the algebra studied by advanced college-level students. And, although the prerequisites assumed of our readers limit our exploration of the development of the more advanced algebraic topics encountered at the graduate and research levels, we provide at least a glimpse of the origins of some of those more advanced topics in the book's final chapters.

SETTING AND EXAMINING THE HISTORICAL PARAMETERS

Nearly five decades before the National Mathematics Advisory Panel issued its report, historian of mathematics, Michael Mahoney, gave a more abstract definition of algebra, or, as he termed it, the "algebraic mode of thought":

> What should be understood as the "algebraic mode of thought"? It has three main characteristics: first, this mode of thought is characterized by the use of an operative symbolism, that is, a symbolism that not only abbreviates words but represents the workings of the combinatory operations, or, in other words, a symbolism with which one operates. Second, precisely because of the central role of combinatory operations, the algebraic mode of thought deals

> with mathematical relations rather than objects. Third, the algebraic mode of thought is free of ontological commitment. . . .In particular, this mode of thought is free of the intuitive ontology of the physical world. Concepts like "space," "dimension," and even "number" are understood in a purely mathematical sense, without reference to their physical interpretation.[5]

Interestingly, Mahoney's first characteristic of algebraic thought as an "operative symbolism"—as well as the discussion of symbolism—is the first of the topics mentioned in the NMAP report. If, however, we believed that an operative symbolism is a necessary characteristic of algebra, this book would not begin before the seventeenth century since, before that time, mathematics was generally carried out in words. Here, we shall argue that symbolism is *not* necessary for algebra, although it has certainly come to characterize it—and, indeed, all of mathematics—over the past three centuries. We shall also argue that, initially, algebra dealt with objects rather than relations and that the beginnings of algebra actually *required* physical interpretations.

The roots of algebra go back thousands of years, as we shall see in the next chapter, but the two earliest texts that serve to define a subject of algebra are the *Arithmetica* of Diophantus (third century CE) and *The Compendious Book on the Calculation of al-Jabr and al-Muqābala* of al-Khwārizmī (ninth century CE). Although neither of these books required physical interpretations of the problems they presented, they did deal with objects rather than relations and neither used any operative symbolism. However, as we shall see below, al-Khwārizmī's book in particular was on the cusp of the change from "physical interpretations" to "abstract number" in the development of algebra. And, although the term "algebra" is absent from the texts both of Diophantus and al-Khwārizmī, it is clear that their major goal was to find unknown numbers that were determined by their relationship to already known numbers, that is, in modern terminology, to solve equations. This is also one of the goals listed in the NMAP report, so it would be difficult to deny that these works exhibit "algebraic thought." Thus, in order to study algebra historically, we need a definition of it somewhat different from that of Mahoney, which applies only to the algebra of the past three centuries.

[5] Mahoney, 1971, pp. 1–2.

It is interesting that school algebra texts today do not even attempt to define their subject. In the eighteenth and nineteenth centuries, however, textbook writers had no such compunction. The standard definition, in fact, was one given by Leonhard Euler in his 1770 textbook, *Elements of Algebra*. Algebra, for Euler, was "the science which teaches how to determine unknown quantities by means of those that are known."[6] He thus articulated explicitly what most of his predecessors had implicitly taken as the meaning of their subject, and we follow his lead here in adopting his definition, at least in the initial stages of this book when we explore how "determining unknowns" was accomplished in different times and places.

Now, there is no denying that, taken literally, Euler's definition of algebra is vague. It is, for example, not immediately clear what constitutes the "quantity" to be determined. Certainly, a "number" is a quantity—however one may define "number"—but is a line segment a "quantity"? Is a vector? Euler was actually clear on this point. "In algebra," he wrote, "we consider only numbers, which represent quantities, without regarding the different kinds of quantity."[7] So, unless a line segment were somehow measured and thus represented by a number, Euler would not have considered it a legitimate unknown of an algebraic equation. Given, however, the close relationship between geometry and what was to evolve into algebra, we would be remiss here not to include line segments as possible unknowns in an equation, regardless of how they may be described, or line segments and areas as "knowns," even if they are not measured. By the time our story has progressed into the nineteenth century, moreover, we shall see that the broadening of the mathematical horizon will make it necessary also to consider vectors, matrices, and other types of mathematical objects as unknowns in an equation.

Besides being vague, Euler's definition, taken literally, is also quite broad. It encompasses what we generally think of as "arithmetic," since the sum of 18 and 43 can be thought of as an "unknown" that can be expressed by the modern equation $x = 18 + 43$. To separate arithmetic from algebra, then, our historical analysis will generally be restricted to efforts to find unknowns that are linked to knowns in a more complicated way than just via an operation. This still leaves room for debate, however,

[6] Euler, 1770/1984, p. 186.
[7] Euler, 1770/1984, p. 2.

as to what actually constitutes an "algebraic" problem. In particular, some of the earliest questions in which unknowns are sought involve what we term proportion problems, that is, problems solved through a version of the "rule of three," namely, if $\frac{a}{b} = \frac{x}{c}$, then $x = \frac{ac}{b}$. These appear in texts from ancient Egypt but also from Mesopotamia, India, China, Islam, and early modern Europe. Such problems are even found, in geometric guise, in classical Greek mathematics. However, al-Khwārizmī and his successors generally did not consider proportion problems in discussing their own science of *al-jabr* and *al-muqābala*. Rather, they preferred to treat them as part of "arithmetic," that is, as a very basic part of the foundation of mathematical learning. In addition, such problems generally arose from real-world situations, and their solutions thus answered real-world questions. It would seem that in ancient times, even the solution of what we would call a linear equation in one variable was part of proportion theory, since such equations were frequently solved using "false position," a method clearly based on proportions. Originally, then, such equations fell outside the concern of algebra, even though they are very much part of algebra now.

Given these historical vagaries, it is perhaps easiest to trace the development of algebra through the search for solutions to what we call quadratic equations. In the "West"—which, for us, will include the modern-day Middle East as far as India in light of what we currently know about the transmission of mathematical thought—a four-stage process can be identified in the history of this part of algebra. The first, *geometric stage* goes back some four millennia to Mesopotamia, where the earliest examples of quadratic equations are geometric in the sense that they ask for the unknown length of a side of a rectangle, for example, given certain relations involving the sides and the area. In general, problems were solved through manipulations of squares and rectangles and in purely geometric terms. Still, Mesopotamian mathematicians were flexible enough to treat quadratic problems not originally set in a geometric context by translating them into their geometric terminology. Mesopotamian methods for solving quadratic problems were also reflected in Greek geometric algebra, whether or not the Greeks were aware of the original context, as well as in some of the earliest Islamic algebraic texts.

Al-Khwārizmī's work, however, marked a definite shift to what may be called the static, equation-solving, *algorithmic stage* of algebra. Although al-Khwārizmī and other Islamic authors justified their methods through geometry—either through Mesopotamian cut-and-paste geometry or through formal Greek geometry—they were interested not in finding *sides* of squares or rectangles but in finding *numbers* that satisfied certain conditions, numbers, in other words, that were not tied to any geometric object. The procedure for solving a quadratic equation for a number is, of course, the same as that for finding the side of a square, but the origin of a more recognizable algebra can be seen as coinciding with this change from the geometric to the algorithmic state, that is, from the quest for finding a geometric object to the search for just an unknown "thing." The solution of cubic equations followed the same path as that of the quadratics, moving from an original geometric stage, as seen initially in the writing of Archimedes (third century BCE) and then later in the work of various medieval Islamic mathematicians, into an algorithmic stage by the sixteenth century.

Interestingly, in India, there is no evidence of an evolution from a geometric stage to an algorithmic one, although the ancient Indians knew how to solve certain problems through the manipulation of squares and rectangles. The earliest written Indian sources that we have containing quadratic equations teach their solution via a version of the quadratic formula. In China, on the other hand, there is no evidence of either geometric or algorithmic reasoning in the solution of quadratic equations. All equations, of whatever degree above the first, were solved through approximation techniques. Still, both Indian and Chinese mathematicians developed numerical algorithms to solve other types of equations, especially indeterminate ones. One of our goals in this book is thus to highlight how each of these civilizations approached what we now classify as algebraic reasoning.

With the introduction of a flexible and operative symbolism in the late sixteenth and seventeenth century by François Viète, Thomas Harriot, René Descartes, and others, algebra entered yet another new stage. It no longer reflected the quest to find merely a numerical solution to an equation but expanded to include complete curves as represented by equations in *two* variables. This stage—marked by the appearance of analytic geometry—may be thought of as the *dynamic stage*, since studying curves as solutions of equations—now termed differential equations—arose in problems about motion.

New symbolism for representing curves also made it possible to translate the complicated geometric descriptions of conic sections that Apollonius had formulated in the third century BCE into brief symbolic equations. In that form, mathematics became increasingly democratic, that is, accessible for mastery to greater numbers of people. This was even true of solving static equations. The verbal solutions of complicated problems, as exemplified in the work of authors like the ninth-century Egyptian Abū Kāmil and the thirteenth-century Italian Leonardo of Pisa, were extremely difficult to follow, especially given that copies of their manuscripts frequently contained errors. The introduction of symbolism, with its relatively simple rules of operation, made it possible for more people to understand mathematics and thus, ultimately, for more mathematics to be created. It also provided a common language that, once adopted, damped regional differences in approach.[8]

Moreover, spurred by Cardano's publication in 1545 of the algorithmic solutions of cubic and quartic equations, the new symbolism enabled mathematicians to pursue the solution of equations of degree higher than four. That quest ultimately redirected algebra from the relatively concrete goal of finding solutions to equations to a more *abstract stage*, in which the study of structures—that is, sets with well-defined axioms for combining two elements—ultimately became paramount. In this changed algebraic environment, groups were introduced in the nineteenth century to aid in the determination of which equations of higher degree were, in fact, solvable by radicals, while determinants, vectors, and matrices were developed to further the study of systems of linear equations, especially when those systems had infinitely many solutions.

Complex numbers also arose initially as a result of efforts to understand the algorithm for solving cubic equations, but subsequently took on a life of their own. Mathematicians first realized that the complex numbers possessed virtually the same properties as the real numbers, namely, the properties of what became known as a field. This prompted the search for other such systems. Given fields of various types, then, it

[8] This is not to say that indigenous techniques and traditions did not persist. Owing to political and cultural mores, for example, Japan and China can be said to have largely maintained indigenous mathematical traditions through the nineteenth century. However, see Hsia, 2009 and Jami, 2012 for information on the introduction of European mathematics into China beginning in the late sixteenth century.

was only natural to look at the analogues of integers in those fields, a step that led ultimately to the notions of rings, modules, and ideals. In yet a different vein, mathematicians realized that complex numbers provided a way of multiplying vectors in the plane. This recognition motivated the nineteenth-century Irish mathematician, William Rowan Hamilton, to seek an analogous generalization for three-dimensional space. Although that problem proved insoluble, Hamilton's pursuits resulted in a four-dimensional system of "generalized numbers," the quaternions, in which the associative law of multiplication held but not the commutative law. Pushing this idea further, Hamilton's successors over the next century developed the even more general notion of algebras, that is, n-dimensional spaces with a natural multiplication.

At the close of the nineteenth century, the major textbooks continued to deem the solution of equations the chief goal of algebra, that is, its main defining characteristic as a mathematical subject matter. The various structures that had been developed were thus viewed as a means to that end. In the opening decades of the twentieth century, however, the hierarchy flipped. The work of the German mathematician, Emmy Noether, as well as her students and mathematical fellow travelers fundamentally reoriented algebra from the more particular and, in some sense, applied solution of equations to the more general and abstract study of structures per se. The textbook, *Moderne Algebra* (1930–1931), by one of those students, Bartel van der Waerden, became the manifesto for this new definition of algebra that has persisted into the twenty-first century.

THE TASK AT HAND

Here, we shall trace the evolution of the algebraic ideas sketched above, delving into some of the many intricacies of the historical record. We shall consider the context in which algebraic ideas developed in various civilizations and speculate, where records do not exist, as to the original reasoning of the developers. We shall see that some of the same ideas appeared repeatedly over time and place and wonder if there were means of transmission from one civilization to another that are currently invisible in the historical record. We shall also observe how mathematicians, once they found solutions to concrete problems, frequently generalized to situations well beyond the original question. Inquiries into these and other issues will allow us to reveal not only the historicity but also the complexity of trying to answer the question, what is algebra?, a question, as we shall see, with different answers for different people in different times and places.

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