

# Contents

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	Acknowledgments	xi
<b>1</b>	<b>Prelude: What Is Algebra?</b>	1
	Why This Book?	3
	Setting and Examining the Historical Parameters	4
	The Task at Hand	10
<b>2</b>	<b>Egypt and Mesopotamia</b>	12
	Proportions in Egypt	12
	Geometrical Algebra in Mesopotamia	17
<b>3</b>	<b>The Ancient Greek World</b>	33
	Geometrical Algebra in Euclid's <i>Elements</i> and <i>Data</i>	34
	Geometrical Algebra in Apollonius's <i>Conics</i>	48
	Archimedes and the Solution of a Cubic Equation	53
<b>4</b>	<b>Later Alexandrian Developments</b>	58
	Diophantine Preliminaries	60
	A Sampling from the <i>Arithmetica</i> : The First Three Greek Books	63
	A Sampling from the <i>Arithmetica</i> : The Arabic Books	68
	A Sampling from the <i>Arithmetica</i> : The Remaining Greek Books	73
	The Reception and Transmission of the <i>Arithmetica</i>	77
<b>5</b>	<b>Algebraic Thought in Ancient and Medieval China</b>	81
	Proportions and Linear Equations	82
	Polynomial Equations	90
	Indeterminate Analysis	98
	The Chinese Remainder Problem	100

<b>6</b>	<b>Algebraic Thought in Medieval India</b>	105
	Proportions and Linear Equations	107
	Quadratic Equations	109
	Indeterminate Equations	118
	Linear Congruences and the Pulverizer	119
	The Pell Equation	122
	Sums of Series	126
<b>7</b>	<b>Algebraic Thought in Medieval Islam</b>	132
	Quadratic Equations	137
	Indeterminate Equations	153
	The Algebra of Polynomials	158
	The Solution of Cubic Equations	165
<b>8</b>	<b>Transmission, Transplantation, and Diffusion in the Latin West</b>	174
	The Transplantation of Algebraic Thought in the Thirteenth Century	178
	The Diffusion of Algebraic Thought on the Italian Peninsula and Its Environs from the Thirteenth Through the Fifteenth Centuries	190
	The Diffusion of Algebraic Thought and the Development of Algebraic Notation outside of Italy	204
<b>9</b>	<b>The Growth of Algebraic Thought in Sixteenth-Century Europe</b>	214
	Solutions of General Cubics and Quartics	215
	Toward Algebra as a General Problem-Solving Technique	227
<b>10</b>	<b>From Analytic Geometry to the Fundamental Theorem of Algebra</b>	247
	Thomas Harriot and the Structure of Equations	248
	Pierre de Fermat and the <i>Introduction to Plane and Solid Loci</i>	253
	Albert Girard and the Fundamental Theorem of Algebra	258

	René Descartes and <i>The Geometry</i>	261
	Johann Hudde and Jan de Witt, Two Commentators on <i>The Geometry</i>	271
	Isaac Newton and the <i>Arithmetica universalis</i>	275
	Colin Maclaurin's <i>Treatise of Algebra</i>	280
	Leonhard Euler and the Fundamental Theorem of Algebra	283
<b>11</b>	<b>Finding the Roots of Algebraic Equations</b>	289
	The Eighteenth-Century Quest to Solve Higher-Order Equations Algebraically	290
	The Theory of Permutations	300
	Determining Solvable Equations	303
	The Work of Galois and Its Reception	310
	The Many Roots of Group Theory	317
	The Abstract Notion of a Group	328
<b>12</b>	<b>Understanding Polynomial Equations in <math>n</math> Unknowns</b>	335
	Solving Systems of Linear Equations in $n$ Unknowns	336
	Linearly Transforming Homogeneous Polynomials in $n$ Unknowns: Three Contexts	345
	The Evolution of a Theory of Matrices and Linear Transformations	356
	The Evolution of a Theory of Invariants	366
<b>13</b>	<b>Understanding the Properties of "Numbers"</b>	381
	New Kinds of "Complex" Numbers	382
	New Arithmetics for New "Complex" Numbers	388
	What Is Algebra?: The British Debate	399
	An "Algebra" of Vectors	408
	A Theory of Algebras, Plural	415
<b>14</b>	<b>The Emergence of Modern Algebra</b>	427
	Realizing New Algebraic Structures Axiomatically	430
	The Structural Approach to Algebra	438
	References	449
	Index	477

# 1

## Prelude: What Is Algebra?

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What is algebra? It is a question to which a high school student will give one answer, a college student majoring in mathematics another, and a professor who teaches graduate courses and conducts algebraic research a third. The educated “layperson,” on the other hand, might simply grimace while retorting, “Oh, I never did well in mathematics. Wasn’t algebra all of that  $x$  and  $y$  stuff that I could never figure out?” This ostensibly simple question, then, apparently has a number of possible answers. What do the “experts” say?

On 18 April 2006, the National Mathematics Advisory Panel (NMAP) within the US Department of Education was established by executive order of then President George W. Bush to advise him, as well as the Secretary of Education, on means to “foster greater knowledge of and improved performance in mathematics among American students.”<sup>1</sup> Among the panel’s charges was to make recommendations on “the critical skills and skill progressions for students to acquire competence in algebra and readiness for higher levels of mathematics.” Why should competence in algebra have been especially singled out?

When it issued its final report in March 2008, the panel stated that “a strong grounding in high school mathematics through Algebra II or higher correlates powerfully with access to college, graduation from college, and earning in the top quartile of income from employment.”<sup>2</sup> Furthermore, it acknowledged that “although our students encounter difficulties with many aspects of mathematics, many observers of

<sup>1</sup> US Dept. of Education, 2008, p. 71. The next quotation is also found here.

<sup>2</sup> US Dept. of Education, 2008, p. xii. For the next two quotations, see pp. xiii and 16, respectively.

## 2 Chapter 1

educational policy see Algebra as a central concern.” The panel had thus sought to determine how best to prepare students for entry into algebra and, since algebra was of such concern, it had first to come to terms with the question, what is the essential content of school algebra? In answer, it identified the following as the major topics: symbols and expressions, linear equations, quadratic equations, functions, the algebra of polynomials, and combinatorics and finite probability. Of course, each of these topics encompasses several subtopics. For example, the “algebra of polynomials” includes complex numbers and operations, the fundamental theorem of algebra, and Pascal’s triangle. Interestingly, the panel mentioned “logarithmic functions” and “trigonometric functions” under the topic of “functions” but made no explicit mention of analytic geometry except in the special case of graphs of quadratic functions. Although the details of the panel’s list might prompt these and other quibbles, it nevertheless gives some idea of what high school students, in the United States at least, generally study—or should study—under the rubric of “algebra.”

These topics, however, constitute “school algebra.” What about algebra at the college level? Most courses entitled “college algebra” in the United States simply revisit the aforementioned topics, sometimes going into slightly greater depth than is expected in high school. Courses for mathematics majors—entitled “modern algebra” or “abstract algebra”—are quite another matter, however. They embrace totally different topics: groups, rings, fields, and, often, Galois theory. Sometimes such courses also include vectors, matrices, determinants, and algebras (where the latter is a technical term quite different from the broad subject under consideration here).

And then there is algebra at the graduate and research levels. Graduate students may take courses in commutative or noncommutative algebra, representation theory, or Lie theory, while research mathematicians styled “algebraists” may deal with topics like “homological functors on modules,” “algebraic coding theory,” “regular local rings,” or any one of hundreds of topics listed in the American Mathematical Society’s “Mathematics Subject Classification.” How do all of these subjects at all of these levels of sophistication fit together to constitute something called “algebra”? Before addressing this question, we might first ask why we need *this* book about it?

## WHY THIS BOOK?

To be sure, the historical literature already includes several more or less widely ranging books on the history of algebra that are targeted, like the present book, at those with a background equivalent to a college major in mathematics;<sup>3</sup> a recent “popular” book assumes even less in the way of mathematical prerequisites.<sup>4</sup> Most in the former group, however, are limited either in the eras covered or in geographical reach, while that in the latter has too many errors of fact and interpretation to stand unchallenged. *This* book thus grew out of a shared realization that the time was ripe for a history of algebra that told the broader story by incorporating new scholarship on the diverse regions within which algebraic thought developed and by tracing the major themes into the early twentieth century with the advent of the so-called “modern algebra.”

We also believe that this is a story very much worth telling, since it is a history very much worth knowing. Using the history of algebra, teachers of the subject, either at the school or at the college level, can increase students’ overall understanding of the material. The “logical” development so prevalent in our textbooks is often sterile because it explains neither why people were interested in a particular algebraic topic in the first place nor why our students should be interested in that topic today. History, on the other hand, often demonstrates the reasons for both. With an understanding of the historical development of algebra, moreover, teachers can better impart to their students an appreciation that algebra is not arbitrary, that it is not created “full-blown” by fiat. Rather, it develops at the hands of people who need to solve vital problems, problems the solutions of which merit understanding. Algebra has been and is being created in many areas of the world, with the same solution often appearing in disparate times and places.

And this is neither a story nor a history limited to school students and their teachers. College-level mathematics students and their

<sup>3</sup> In fact, the prerequisites for reading the first ten chapters are little more than a solid high school mathematics education. The more general histories of algebra include van der Waerden, 1985; Scholz, 1990; Bashmakova and Smirnova, 2000; and Cooke, 2008, while the more targeted include Nový, 1973; Sesiano, 1999 and 2009; Kleiner, 2007; and Stedall, 2011.

<sup>4</sup> Derbyshire, 2006.

professors should also know the roots of the algebra they study. With an understanding of the historical development of the field, professors can stimulate their students to master often complex notions by motivating the material through the historical questions that prompted its development. In absorbing the idea, moreover, that people struggled with many important mathematical ideas before finding their solutions, that they frequently could not solve problems entirely, and that they consciously left them for their successors to explore, students can better appreciate the mathematical endeavor and its shared purpose. To paraphrase the great seventeenth- and early eighteenth-century English mathematician and natural philosopher, Sir Isaac Newton, mathematicians have always seen farther by “standing on the shoulders” of those who came before them.

One of our goals in the present book is thus to show how—in often convoluted historical twists and turns—the deeper and deeper consideration of some of the earliest algebraic topics—those generally covered in schools—ultimately led mathematicians to discover or invent the ideas that constitute much of the algebra studied by advanced college-level students. And, although the prerequisites assumed of our readers limit our exploration of the development of the more advanced algebraic topics encountered at the graduate and research levels, we provide at least a glimpse of the origins of some of those more advanced topics in the book’s final chapters.

## SETTING AND EXAMINING THE HISTORICAL PARAMETERS

Nearly five decades before the National Mathematics Advisory Panel issued its report, historian of mathematics, Michael Mahoney, gave a more abstract definition of algebra, or, as he termed it, the “algebraic mode of thought”:

What should be understood as the “algebraic mode of thought”? It has three main characteristics: first, this mode of thought is characterized by the use of an operative symbolism, that is, a symbolism that not only abbreviates words but represents the workings of the combinatory operations, or, in other words, a symbolism with which one operates. Second, precisely because of the central role of combinatory operations, the algebraic mode of thought deals

with mathematical relations rather than objects. Third, the algebraic mode of thought is free of ontological commitment. . . . In particular, this mode of thought is free of the intuitive ontology of the physical world. Concepts like “space,” “dimension,” and even “number” are understood in a purely mathematical sense, without reference to their physical interpretation.<sup>5</sup>

Interestingly, Mahoney’s first characteristic of algebraic thought as an “operative symbolism”—as well as the discussion of symbolism—is the first of the topics mentioned in the NMAP report. If, however, we believed that an operative symbolism is a necessary characteristic of algebra, this book would not begin before the seventeenth century since, before that time, mathematics was generally carried out in words. Here, we shall argue that symbolism is *not* necessary for algebra, although it has certainly come to characterize it—and, indeed, all of mathematics—over the past three centuries. We shall also argue that, initially, algebra dealt with objects rather than relations and that the beginnings of algebra actually *required* physical interpretations.

The roots of algebra go back thousands of years, as we shall see in the next chapter, but the two earliest texts that serve to define a subject of algebra are the *Arithmetica* of Diophantus (third century CE) and *The Compendious Book on the Calculation of al-Jabr and al-Muqābala* of al-Khwārizmī (ninth century CE). Although neither of these books required physical interpretations of the problems they presented, they did deal with objects rather than relations and neither used any operative symbolism. However, as we shall see below, al-Khwārizmī’s book in particular was on the cusp of the change from “physical interpretations” to “abstract number” in the development of algebra. And, although the term “algebra” is absent from the texts both of Diophantus and al-Khwārizmī, it is clear that their major goal was to find unknown numbers that were determined by their relationship to already known numbers, that is, in modern terminology, to solve equations. This is also one of the goals listed in the NMAP report, so it would be difficult to deny that these works exhibit “algebraic thought.” Thus, in order to study algebra historically, we need a definition of it somewhat different from that of Mahoney, which applies only to the algebra of the past three centuries.

<sup>5</sup> Mahoney, 1971, pp. 1–2.



## 6 Chapter 1

It is interesting that school algebra texts today do not even attempt to define their subject. In the eighteenth and nineteenth centuries, however, textbook writers had no such compunction. The standard definition, in fact, was one given by Leonhard Euler in his 1770 textbook, *Elements of Algebra*. Algebra, for Euler, was “the science which teaches how to determine unknown quantities by means of those that are known.”<sup>6</sup> He thus articulated explicitly what most of his predecessors had implicitly taken as the meaning of their subject, and we follow his lead here in adopting his definition, at least in the initial stages of this book when we explore how “determining unknowns” was accomplished in different times and places.

Now, there is no denying that, taken literally, Euler’s definition of algebra is vague. It is, for example, not immediately clear what constitutes the “quantity” to be determined. Certainly, a “number” is a quantity—however one may define “number”—but is a line segment a “quantity”? Is a vector? Euler was actually clear on this point. “In algebra,” he wrote, “we consider only numbers, which represent quantities, without regarding the different kinds of quantity.”<sup>7</sup> So, unless a line segment were somehow measured and thus represented by a number, Euler would not have considered it a legitimate unknown of an algebraic equation. Given, however, the close relationship between geometry and what was to evolve into algebra, we would be remiss here not to include line segments as possible unknowns in an equation, regardless of how they may be described, or line segments and areas as “knowns,” even if they are not measured. By the time our story has progressed into the nineteenth century, moreover, we shall see that the broadening of the mathematical horizon will make it necessary also to consider vectors, matrices, and other types of mathematical objects as unknowns in an equation.

Besides being vague, Euler’s definition, taken literally, is also quite broad. It encompasses what we generally think of as “arithmetic,” since the sum of 18 and 43 can be thought of as an “unknown” that can be expressed by the modern equation  $x = 18 + 43$ . To separate arithmetic from algebra, then, our historical analysis will generally be restricted to efforts to find unknowns that are linked to knowns in a more complicated way than just via an operation. This still leaves room for debate, however,

<sup>6</sup> Euler, 1770/1984, p. 186.

<sup>7</sup> Euler, 1770/1984, p. 2.

as to what actually constitutes an “algebraic” problem. In particular, some of the earliest questions in which unknowns are sought involve what we term proportion problems, that is, problems solved through a version of the “rule of three,” namely, if  $\frac{a}{b} = \frac{x}{c}$ , then  $x = \frac{ac}{b}$ . These appear in texts from ancient Egypt but also from Mesopotamia, India, China, Islam, and early modern Europe. Such problems are even found, in geometric guise, in classical Greek mathematics. However, al-Khwārizmī and his successors generally did not consider proportion problems in discussing their own science of *al-jabr* and *al-muqābala*. Rather, they preferred to treat them as part of “arithmetic,” that is, as a very basic part of the foundation of mathematical learning. In addition, such problems generally arose from real-world situations, and their solutions thus answered real-world questions. It would seem that in ancient times, even the solution of what we would call a linear equation in one variable was part of proportion theory, since such equations were frequently solved using “false position,” a method clearly based on proportions. Originally, then, such equations fell outside the concern of algebra, even though they are very much part of algebra now.

Given these historical vagaries, it is perhaps easiest to trace the development of algebra through the search for solutions to what we call quadratic equations. In the “West”—which, for us, will include the modern-day Middle East as far as India in light of what we currently know about the transmission of mathematical thought—a four-stage process can be identified in the history of this part of algebra. The first, *geometric stage* goes back some four millennia to Mesopotamia, where the earliest examples of quadratic equations are geometric in the sense that they ask for the unknown length of a side of a rectangle, for example, given certain relations involving the sides and the area. In general, problems were solved through manipulations of squares and rectangles and in purely geometric terms. Still, Mesopotamian mathematicians were flexible enough to treat quadratic problems not originally set in a geometric context by translating them into their geometric terminology. Mesopotamian methods for solving quadratic problems were also reflected in Greek geometric algebra, whether or not the Greeks were aware of the original context, as well as in some of the earliest Islamic algebraic texts.

Al-Khwārizmī’s work, however, marked a definite shift to what may be called the static, equation-solving, *algorithmic stage* of algebra. Although

al-Khwārizmī and other Islamic authors justified their methods through geometry—either through Mesopotamian cut-and-paste geometry or through formal Greek geometry—they were interested not in finding *sides* of squares or rectangles but in finding *numbers* that satisfied certain conditions, numbers, in other words, that were not tied to any geometric object. The procedure for solving a quadratic equation for a number is, of course, the same as that for finding the side of a square, but the origin of a more recognizable algebra can be seen as coinciding with this change from the geometric to the algorithmic state, that is, from the quest for finding a geometric object to the search for just an unknown “thing.” The solution of cubic equations followed the same path as that of the quadratics, moving from an original geometric stage, as seen initially in the writing of Archimedes (third century BCE) and then later in the work of various medieval Islamic mathematicians, into an algorithmic stage by the sixteenth century.

Interestingly, in India, there is no evidence of an evolution from a geometric stage to an algorithmic one, although the ancient Indians knew how to solve certain problems through the manipulation of squares and rectangles. The earliest written Indian sources that we have containing quadratic equations teach their solution via a version of the quadratic formula. In China, on the other hand, there is no evidence of either geometric or algorithmic reasoning in the solution of quadratic equations. All equations, of whatever degree above the first, were solved through approximation techniques. Still, both Indian and Chinese mathematicians developed numerical algorithms to solve other types of equations, especially indeterminate ones. One of our goals in this book is thus to highlight how each of these civilizations approached what we now classify as algebraic reasoning.

With the introduction of a flexible and operative symbolism in the late sixteenth and seventeenth century by François Viète, Thomas Harriot, René Descartes, and others, algebra entered yet another new stage. It no longer reflected the quest to find merely a numerical solution to an equation but expanded to include complete curves as represented by equations in *two* variables. This stage—marked by the appearance of analytic geometry—may be thought of as the *dynamic stage*, since studying curves as solutions of equations—now termed differential equations—arose in problems about motion.

New symbolism for representing curves also made it possible to translate the complicated geometric descriptions of conic sections that Apollonius had formulated in the third century BCE into brief symbolic equations. In that form, mathematics became increasingly democratic, that is, accessible for mastery to greater numbers of people. This was even true of solving static equations. The verbal solutions of complicated problems, as exemplified in the work of authors like the ninth-century Egyptian Abū Kāmil and the thirteenth-century Italian Leonardo of Pisa, were extremely difficult to follow, especially given that copies of their manuscripts frequently contained errors. The introduction of symbolism, with its relatively simple rules of operation, made it possible for more people to understand mathematics and thus, ultimately, for more mathematics to be created. It also provided a common language that, once adopted, damped regional differences in approach.<sup>8</sup>

Moreover, spurred by Cardano's publication in 1545 of the algorithmic solutions of cubic and quartic equations, the new symbolism enabled mathematicians to pursue the solution of equations of degree higher than four. That quest ultimately redirected algebra from the relatively concrete goal of finding solutions to equations to a more *abstract stage*, in which the study of structures—that is, sets with well-defined axioms for combining two elements—ultimately became paramount. In this changed algebraic environment, groups were introduced in the nineteenth century to aid in the determination of which equations of higher degree were, in fact, solvable by radicals, while determinants, vectors, and matrices were developed to further the study of systems of linear equations, especially when those systems had infinitely many solutions.

Complex numbers also arose initially as a result of efforts to understand the algorithm for solving cubic equations, but subsequently took on a life of their own. Mathematicians first realized that the complex numbers possessed virtually the same properties as the real numbers, namely, the properties of what became known as a field. This prompted the search for other such systems. Given fields of various types, then, it

<sup>8</sup> This is not to say that indigenous techniques and traditions did not persist. Owing to political and cultural mores, for example, Japan and China can be said to have largely maintained indigenous mathematical traditions through the nineteenth century. However, see Hsia, 2009 and Jami, 2012 for information on the introduction of European mathematics into China beginning in the late sixteenth century.

was only natural to look at the analogues of integers in those fields, a step that led ultimately to the notions of rings, modules, and ideals. In yet a different vein, mathematicians realized that complex numbers provided a way of multiplying vectors in the plane. This recognition motivated the nineteenth-century Irish mathematician, William Rowan Hamilton, to seek an analogous generalization for three-dimensional space. Although that problem proved insoluble, Hamilton's pursuits resulted in a four-dimensional system of "generalized numbers," the quaternions, in which the associative law of multiplication held but not the commutative law. Pushing this idea further, Hamilton's successors over the next century developed the even more general notion of algebras, that is,  $n$ -dimensional spaces with a natural multiplication.

At the close of the nineteenth century, the major textbooks continued to deem the solution of equations the chief goal of algebra, that is, its main defining characteristic as a mathematical subject matter. The various structures that had been developed were thus viewed as a means to that end. In the opening decades of the twentieth century, however, the hierarchy flipped. The work of the German mathematician, Emmy Noether, as well as her students and mathematical fellow travelers fundamentally reoriented algebra from the more particular and, in some sense, applied solution of equations to the more general and abstract study of structures per se. The textbook, *Moderne Algebra* (1930–1931), by one of those students, Bartel van der Waerden, became the manifesto for this new definition of algebra that has persisted into the twenty-first century.

## THE TASK AT HAND

Here, we shall trace the evolution of the algebraic ideas sketched above, delving into some of the many intricacies of the historical record. We shall consider the context in which algebraic ideas developed in various civilizations and speculate, where records do not exist, as to the original reasoning of the developers. We shall see that some of the same ideas appeared repeatedly over time and place and wonder if there were means of transmission from one civilization to another that are currently invisible in the historical record. We shall also observe how mathematicians, once they found solutions to concrete problems,

frequently generalized to situations well beyond the original question. Inquiries into these and other issues will allow us to reveal not only the historicity but also the complexity of trying to answer the question, what is algebra?, a question, as we shall see, with different answers for different people in different times and places.

## Index

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- abbacus schools (*see also* libri d'abbaco), 191, 235
- Abel, Niels Henrik, 299–300, 309–310
- abelian groups, 319; fundamental theorem of, 321
- Abraham bar Hiyya of Barcelona (or Savasorda), 137n6, 178n6, 181n14
- Abu Bekr, 137, 181n14
- Abū Kāmil ibn Aslam, 9, 147–157, 172–173
- academic competition in Renaissance Italy, 216–219
- Academy of Sciences, Paris (or Académie des Sciences), 292, 299; and the reception of Galois's work, 311n43
- A'h-mose, 13
- Alberti, Leon Battista, 200–201
- Alexander the Great, 58, 105
- Alexandria, 58–60, 133; algebra in (*see* algebra, in Alexandria); Muslim conquest of, 79–80
- algebra: in Alexandria, 58–80; in ancient Greece, 7–8, 33–57; in China, 7, 8, 81–104, 335; commutative (*see* commutative algebra); definitions of, 1–2, 5–7, 36, 236–238, 334, 399–408, 417; in Egypt, 7, 12–16, 32; in eighteenth-century Europe, 280–288; geometrical, 17–30, 34–53, 57, 139, 238–240, 250; in India, 7, 8, 105–131, 335; in medieval Europe, 178–213; in medieval Islam, 7–8, 132–173, 335; in Mesopotamia, 7–8, 17–32, 137, 139; modern (*see* modern algebra); polynomial (*see* polynomial algebra); in seventeenth-century Europe, 248–280; in sixteenth-century Europe, 214–246, 335; stages in development of, 7–10, 20, 139, 267, 287–288; structural approach to, 333, 428, 433, 437, 438–447
- algebraic curves, 339–340, 342
- algebraic function fields, 399
- algebraic geometry, 399, 423
- algebraic integers, 394–399
- algebraic number field, 394–395, 397, 433
- algebraic notation (*see* symbolism in algebra)
- algebras, 2, 10, 358, 415–426, 428, 442, 444; axiomatization of, 437
- analysis and synthesis, 78, 237–238
- analytic geometry, 49, 53, 253–258, 262–268, 269–274, 280
- Apollonius, 9, 48, 58, 133, 253, 265; *Conics*, 48–53, 78, 167, 247, 257–258, 279–280; as reinterpreted in the work of Fermat, 253–255
- application of areas, 42–43, 48
- Archimedes, 53–54, 58, 133, 258; *The Sphere and the Cylinder*, 53–57; work of on cubic equations, 53–57, 165–166, 168
- Archytas, 33
- Aristotle, 34
- Arithmetica* of Diophantus, 60–80, 156–158, 187n27, 227–228, 229–230, 233–235, 237–238, 240, 335; cubic equations in the, 70–72; indeterminate equations in the, 64–68, 335; linear equations in the, 63–64; notational conventions in the, 61–62; powers of the unknown in the, 61, 69, 158–159; quadratic equations in the, 64–68, 72–77; quartic equations in the, 74–77; reception and transmission of the, 77–80
- arithmetic progression, 30–31, 111, 159–160
- Aronhold, Siegfried, 367, 374–377
- Artin, Emil, 443, 444
- Āryabhaṭīya, 106, 108, 113, 116, 121; *Āryabhaṭīya*, 106–108, 111–112, 116, 119, 126–127
- Arzelà, Cesare, 324n80
- Ashoka, 105
- Autolytus, 33
- axiomatization: of an algebra, 437; of a field, 434–437; of a group by Weber, 329–330; Hilbert's call for, 429–430; of a ring, 433–434, 440–441; of a vector space by Peano, 430–432
- Babbage, Charles, 401, 404
- Bachmann, Paul, 324n81

- back-reckoning, 74–75  
Bagio the Elder, 194  
Banach, Stefan, 432  
*Baudhāyana-sulba-sūtra* (see *Sulba-sūtras*)  
Bellavitis, Giusto, 410–411  
Benedetto, Maestro, 197–200  
*Berlin Papyrus*, 15  
Bernoulli, Daniel, 345  
Bernoulli, Johann, 283, 345  
Bernoulli, Nicolaus, 284, 286  
Bertrand, Joseph, 315  
Bezout, Étienne, 292–294, 343  
Bhāskara I, 106–108, 112, 119  
Bhāskara II, 113–114, 116; *Bīja-ganita*, 113–116, 118, 120–121, 124–126  
binary forms, 332, 351–355, 368–378  
binomial theorem, 90–91, 162–164  
biquadratic reciprocity, law of, 386–388  
biquaternions (see also quaternions, complex), 408  
al-Bīrūnī, Abū Rayhān, 135–136  
Bombelli, Rafael, 62, 228; *L'Algebra*, 229–235, 381  
Boole, George, 367–368, 369; work of on linear transformation, 350–353  
Borchardt, Carl, 320, 349  
Borrel, Jean (also known as Johannes Buteo), 212, 235–236  
Brahmagupta, 106, 108–109, 112–113, 119–124  
Bravais, Auguste, 328n93  
Brioschi, Francesco, 344  
Bruno, Francesco Faà di, 344  
Burali-Forti, Cesare, 432  
Burnside, William, 333, 442n26  
Bush, George W., 1  
Buteo, Johannes (see Borrel, Jean)
- Cambridge Analytical Society, 401, 404  
*Cambridge Mathematical Journal*, 404  
Cantor, Georg, 430  
Cardano, Girolamo, 9, 216, 218–227, 233, 235, 245–246; *Ars magna*, 216, 219–227, 228, 231–232, 233, 235, 251, 259, 276, 282, 289, 381; and the solution of the cubic equation, 219–222, 231–232, 233, 276, 282; and the solution of the quartic equation, 222–225, 227  
Carolingian Empire, 175–176  
Cartan, Élie, 425  
Catherine I of Russia, 291  
Cauchy, Augustin-Louis, 300–301; determinants in the work of, 343–344, 347–349, 353, 357, 361; permutations in the work of, 300–303; and the principal axis problem, 347–349  
Cayley, Arthur, 321–322, 344, 353, 363–364, 366–369, 375, 378, 408, 411; and the Cayley-Hermite problem, 357; work of on the theory of groups, 321–325; work of on the theory of invariants, 366–371; work of on the theory of matrices, 356–360, 428  
Cayley-Hermite problem, 357–358, 364–366  
Cayley's theorem, 325  
chain conditions on ideals (see ideals, chain conditions on)  
Chandra Gupta II, 105  
Chandragupta Maurya, 105  
characteristic (of a field) (see fields, characteristic of)  
characteristic polynomial, 348, 359–363  
Charlemagne, 175–176  
chemistry and group theory, 328n93  
Chevalier, Auguste, 311  
China, algebra in (see algebra, in China)  
Chinese remainder problem, 82, 100–104  
Chrysippus, 34  
Chuquet, Nicolas, 204–205, 230n37, 233n44  
Citrahānu, 116–118  
Clavius, Christoph, 213  
Clebsch, Alfred, 367, 376–377, 413  
Clifford, William Kingdon, 420–421  
Cockle, James, 408  
Colson, John, 282  
Commandino, Federico, 237  
commutative algebra, 423  
completing the square, 227; in the *Elements* of Euclid, 39–40; in Indian mathematics, 109–111, 114; in Islamic mathematics, 40–41; in Mesopotamian mathematics, 20–22, 24–25  
complex numbers, 9–10, 225–227, 230–233, 252, 260–261, 270, 381, 386–388; ideal (see ideal complex numbers); in the work of Euler, 382–384  
congruences, 100–104, 119–121, 385  
conic sections (see also ellipse; hyperbola; parabola), 48–49, 257–258, 267, 272–274, 279–280  
cossic algebra, 205–212  
counting boards, 85–86, 93, 95, 97–98  
cyclotomic integers, 390  
Cramer, Gabriel, 339–342  
Cramer's paradox, 342, 414



- Cramer's rule, 341–342  
Crelle, Auguste, 320  
Crelle's *Journal* (see *Journal für die reine und angewandte Mathematik*)  
cubic equations, 245–246, 291–292, 295–297;  
in Islamic mathematics, 165–172; and the solution of  $x^3 + px = q$ , 215–216, 218–219, 241–244, 251–252; in the work of Archimedes, 53–57, 165–166, 167–168; in the work of Bombelli, 230–233; in the work of Cardano, 219–222, 231–232, 233, 276, 282; in the work of Descartes, 267–268, 269–270; in the work of Diophantus, 70–72; in the work of Euler, 382–384; in the work of Fermat, 258; in the work of Girard, 258–259; in the work of Harriot, 251–252; in the work of Leonardo of Pisa, 185–186; in the work of Maclaurin, 282; in the work of the maestri d'abbaco, 192–194, 195–196; in the work of Nuñez, 205; in the work of Pacioli, 214–215; in the work of Viète, 241–244, 245  
cubic reciprocity, law of, 388  
cyclotomic equations, 303–309, 313  
cyclotomic integers, 390, 391–394, 396  
  
d'Alembert, Jean le Rond, 285, 345  
Darboux, Gaston, 327  
Dedekind, Richard, 324, 380, 423, 432; work of on algebraic integers, 394–399; work of on fields, 396–397  
Dedekind domain, 398n43  
de la Roche, Étienne, 205, 235  
del Ferro, Scipione, 215–217, 219  
della Nave, Annibale, 216, 219  
de Moivre, Abraham, 290–291  
De Morgan, Augustus, 403–404, 415  
Descartes, René, 8, 245n74, 248, 261–262, 280, 338; *La Géométrie* (*The Geometry*), 248, 261, 262–272, 274, 287, 335; rule of signs of, 269, 275–276  
De Séguier, Jean-Armand, 333–334  
determinants, 2, 9, 318, 335–338, 376, 427;  
in the work of Cauchy, 343–344, 347–349, 353, 357, 361; in the work of Sylvester, 344  
Deutsche Mathematiker-Vereinigung, 433n8  
Dickson, Leonard Eugene, 333n107, 437  
Diophantus of Alexandria (see also *Arithmetica* of Diophantus), 7, 60, 134  
Dirichlet, Peter-Lejeune, 320, 389, 394  
discriminant, 180, 224n18, 297, 318, 332, 351–353, 368–369  
  
*Disquisitiones arithmeticae* (see also Gauss, Carl Friedrich): congruence, definition of given in, 385; quadratic forms in, 318–321, 353–355, 357; quadratic reciprocity, proof of law of in, 385–386; solution of cyclotomic equations in, 303–309  
division algebra, 406, 420n98  
Dodgson, Charles, 415  
  
Egypt, algebra in (see algebra, in Egypt)  
eigenvalues, 347, 349  
eigenvectors, 347, 349  
Eisenstein, Gotthold, 355, 357, 366, 373–374, 388, 394  
*Elements* of Euclid, 34–46, 167, 177, 253, 381;  
Book I, 42–43; Book II, 34–42, 46, 109, 147–148, 164; Book VI, 42–46, 48, 153; quadratic equations in the, 35–48  
elimination, theory of, 87, 88, 89n15, 95–98, 99, 209–212, 276, 292–294, 343, 344, 345n17, 350–353  
Elliott, Edwin Bailey, 372  
ellipse, Apollonian definition of, 51, 247, 279  
Ellis, Robert, 404  
Engel, Friedrich, 413  
Eratosthenes, 58  
*Erlanger Programm*, 326  
Euclid (see also *Elements* of Euclid), 34, 38, 58, 133, 147–148, 153; *Data*, 46–48, 167  
Euclidean algorithm, 104, 119–121  
Eudemos of Rhodes, 33  
Eudoxus, 33  
Euler, Leonhard, 6, 345, 357, 385, 400; and Cramer's paradox, 342, 414; *Elements of Algebra*, 6; and the fundamental theorem of algebra, 283–287; and the solution of polynomial equations, 291–292; work of on complex numbers, 382–384; work of on Fermat's last theorem, 381–384, 386  
excess and deficit, method of, 83–87  
exegetics, 237–238  
exterior algebra, 413  
  
factor theorem, 269  
false position: in Chinese mathematics, 83–87; in Egyptian mathematics, 15–16; in Mesopotamian mathematics, 19–20  
Fermat, Pierre de, 248, 252–253, 258, 261; and analytic geometry, 253–254, 262, 265, 280; as interpreted by Hudde and

- Fermat, Pierre de (*continued*)  
Van Schooten, 271–274; *Introduction to Plane and Solid Loci*, 253, 254–258; reinterpretation of Apollonius's work by, 253–255
- Fermat's last theorem, 71, 381–384, 386, 388–399, 428
- Fermat's little theorem, 388n16
- Ferrari, Ludovico, 218–219, 220, 223–224, 245, 252
- fields, 2, 9; axiomatization of, 434–437; characteristic of, 435, 437; real (*see* real fields); in the work of Dedekind, 396–397; in the work of Galois, 312, 396–397; in the work of Steinitz, 397, 434–437, 443–444
- Fini, Mordecai, 173, 197n58
- Fior, Antonio Maria, 217–218
- Fontana, Niccolò (also known as Niccolò Tartaglia), 218–219, 235, 245
- Forcadel, Pierre, 212, 236
- forms, theory of (*see also* invariants, theory of), 344n17, 349–350; Boole's work in the, 350–353, 367; Cauchy's work in the, 347–349; Gauss's work in the, 318–319, 320–321, 353–355, 357; Frobenius's work in the, 363–366; Lagrange's work in the, 345–347; Sylvester's work in the, 349; Weierstrass's work in the, 361–363
- Fourier, Joseph, 311n43
- Fraenkel, Adolf, 433–434, 437
- Francesca, Piero della, 200
- Frederick II, 184–185
- Frend, William, 400–401
- Frobenius, Georg, 330, 349, 442n26; work of on the Cayley-Hermite problem, 364–366; work of in the theory of forms, 363–366; work of on the theory of matrices, 414–415, 428
- fundamental theorem of algebra, 250, 258–261, 267–268, 276, 281–287, 349
- fundamental theorem of arithmetic, 387
- Galen of Pergamum, 59
- Galilei, Galileo, 247, 279
- Galois, Évariste, 310–315, 382; permutations in the work of, 312–314; reception of the work of at the Académie des Sciences, 311n43; work of on field theory, 312, 396–397; work of on groups, 312–313, 427
- Galois group, definition of, 313
- Galois theory, 2, 315–316
- Gauss, Carl Friedrich, 287; solution of cyclotomic equations by, 303–309; work of on the law of biquadratic reciprocity, 386–388; work of on the law of quadratic reciprocity, 385–386; work of on linear transformation, 353–355; work of on quadratic forms, 318–319, 320–321, 353–355, 357
- Gaussian elimination, 87, 88, 89n15, 99
- Gaussian integers, 386–388
- generic reasoning, 360–361, 362, 363–364
- geometrical algebra (*see* algebra, geometrical)
- Gerardi, Paolo, 192–194
- Gerard of Cremona, 177
- Gerbert of Aurillac (later Pope Sylvester II), 176–177
- Germain, Sophie, 388–389
- Gibbs, Josiah Willard, 410
- Gilio of Siena, 194
- Giovanni di Bartolo, 197
- Girard, Albert, 258–261, 283, 343n12
- Goldbach, Christian, 382
- Golius, Jacobus, 262, 266
- Gordan, Paul, 367, 378, 438
- Gosselin, Guillaume, 235–237, 245n74, 248; definition of algebra of, 236
- Grassmann, Hermann, 411, 430; work of on vectors, 411–414
- Graves, Charles, 408
- Graves, John, 407–408
- Greece, algebra in ancient (*see* algebra, in ancient Greece)
- Gregory, Duncan, 404–405
- groups, 2, 9, 289, 302–303, 316–317; abstract definitions of, 322–323, 328–334; axiomatization of by Weber, 329–330; in the work of Cayley, 321–325; in the work of Galois, 312–313, 427; in the work of Jordan, 316–317; theory of, 317–328, 427; of transformations, 327–328
- Hahn, Hans, 432
- Hamilton, William Rowan, 10, 358–359, 360, 366, 402–403, 405; work of on biquaternions, 408; work of on quaternions, 406–407, 415; work of on vectors, 408–410
- Han dynasty, 81
- Hankel, Hermann, 413
- Harriot, Thomas, 8, 248–249, 258, 259, 261, 264, 268, 276; and the structure of equations, 249–252; work of on quartic equations, 251–252
- Heaviside, Oliver, 410

- Hensel, Kurt, 433, 435–436  
Hermite, Charles, 344, 357–358, 394;  
  solution of the quintic equation of, 315;  
  work of on the Cayley-Hermite problem,  
  364, 366  
Hero of Alexandria, 59  
Herschel, John, 401  
Hesse, Otto, 344, 367, 369–370, 373  
higher reciprocity, laws of, 391  
Hilbert, David, 378–380, 423, 429–430,  
  438–439, 440; twenty-three mathematical  
  problems of, 429–430; “Zahlbericht,”  
  433  
Hippocrates of Chios, 33  
history *vs.* heritage, 34–35, 37n6, 53,  
  147–148, 164  
Hölder, Otto, 330–331  
Horner, William, 93  
Horner method, 186n23  
Hudde, Johann, 272  
humanism, 177–178, 235–236, 237  
hundred fowls problem, 98–99, 118,  
  154–155  
Hutton, Charles, 249–250  
Hypatia, 78–79, 174  
hyperbola, Apollonian definition of, 49–51,  
  52, 257  
hypercomplex number systems (*see* algebras)
- ibn Abdun, 137  
ibn al-Haytham, Abū ‘Alī al-Hasan, 164–165,  
  173  
ibn Turk, ‘Abd al-Hamīd ibn Wāsi, 144–145  
ideal complex numbers, 320–321, 390–394,  
  396  
ideals, 397–398, 425n110, 432, 440–442;  
  chain conditions on, 442  
idempotent, 418  
imaginary numbers, 400  
indeterminate equations: in the *Arithmetica* of  
  Diophantus, 64–68, 335; in China, 98–100;  
  in India, 118–119; in Islamic mathematics,  
  153–158  
India, algebra in (*see* algebra, in India)  
inductive proof, 164–165  
infinite descent, method of, 382  
integral domain, 398n43  
International Congress of Mathematicians,  
  429–430  
internationalization of mathematics, 429  
invariants, theory of (*see also* forms, theory  
  of), 318, 372–373, 377–378, 428;  
  Aronhold’s work on the, 374–377; Cayley’s  
  work on the, 366–371; Gordan’s work on  
  the, 378; Hilbert’s work on the, 378–380;  
  Sylvester’s work on the, 353, 367–373,  
  375  
Islam, algebra in medieval (*see* algebra, in  
  medieval Islam)  
Iskur-mansum, 24  
Jacobi, Carl, 344, 349, 388  
Jacopo of Florence, 191–192  
*Jahrbuch über die Fortschritte der Mathematik*,  
  429  
Jayadeva, 124–125  
Jean des Murs (*see* Johannes de Muris)  
Jia Xian, 92  
*Jiuzhang suanshu* (*see* *Nine Chapters on the  
  Mathematical Art*)  
Johannes de Muris (or Jean des Murs),  
  181n14  
John of Palermo, 185  
Jordan, Camille, 316–317, 327; permutations  
  in the work of, 316–317; *Traité des  
  substitutions et des équations algébriques*,  
  316–317, 363  
Jordanus of Nemore, 187–188, 190–191; *De  
  numeris datis*, 187–190, 240n64; quadratic  
  equations in the work of, 188–190  
*Journal für die reine und angewandte  
  Mathematik*, 320  
Jyēsthadeva, 127–130, 165  
Kant, Immanuel, 402  
al-Karajī, Abū Bakr, 156–160, 162, 173  
Kepler, Johannes, 247–248, 279  
Kerala school, 107, 116, 119, 130–131  
Khayyam, Omar (*see* al-Khayyāmī, ‘Umar ibn  
  Ibrāhīm)  
al-Khayyāmī, ‘Umar ibn Ibrāhīm, 166–171,  
  173, 258, 270  
al-Khwārizmī, Muhammad, 5, 7, 138, 158,  
  167–168, 227; *The Compendious Book on the  
  Calculation of al-Jabr and al-Muqābala*, 5,  
  138–147, 172, 177  
Kirkman, Thomas, 408  
Klein, Felix, 326–328, 413, 438–439  
Königsberger, Leo, 438n19  
Kovalevskaya, Sonya, 438n19  
Kronecker, Leopold, 320–321, 360–361,  
  398n45  
Kummer, Ernst, 320; work of on ideal  
  complex numbers, 320, 390–394

- Lacroix, Sylvestre, 401  
Lagrange, Joseph-Louis, 299, 353, 400;  
  *Mécanique analytique*, 345–346;  
  permutations in the work of, 296–299; and  
  the principal axis problem, 345–347, 357;  
  and the solution of polynomial equations,  
  287, 295–298, 303; theorem of,  
  303n28  
Lamé, Gabriel, 389–390  
Laplace, Pierre Simon de, 343, 345  
Lasker, Emmanuel, 440  
law of exponents, 69, 158–161, 183–184, 207,  
  233, 264  
law of inertia, 349  
Legendre, Adrien-Marie, 385, 386, 389  
Leibniz, Gottfried, 283–293, 336–339, 414  
Leonardo of Pisa (see also *Liber abbaci* of  
  Leonardo of Pisa), 9, 172–173, 178–179,  
  190, 275; *Flos*, 185–186; *Liber quadratorum*  
  (or *Book of Squares*), 185–187; work of on  
  quadratic equations, 179–182, 184–187  
l'Hôpital, Marquis de, 336  
*Liber abbaci* of Leonardo of Pisa, 179,  
  184–185, 191, 215, 229; linear equations in  
  the, 179; powers of the unknown in the,  
  183–184; quadratic equations in the,  
  179–182, 184–187  
libri d'abbaco (see also *maestri d'abbaco*),  
  191–200, 229  
Lie, Sophus, 327–328, 424  
Lie group, 424n107, 425n111  
linear algebra, 412–415, 418, 428,  
  445  
linear associative algebras (see algebras)  
linear equations: in the *Arithmetica* of  
  Diophantus, 63–64; in Chinese  
  mathematics, 83–89; in Egyptian  
  mathematics, 14–15; in Indian  
  mathematics, 107–109; in the *Liber abbaci*  
  of Leonardo of Pisa, 179; in Mesopotamian  
  mathematics, 18–19; systems of, 86–89,  
  212, 276, 281–282, 336–344, 348, 414; in  
  the work of Cramer, 339–342; in the work  
  of Viète, 240–241  
linear transformation, 345, 346–347,  
  427–428; Boole's work on, 350–353;  
  Gauss's work on, 353–355; Weierstrass's  
  work on, 361–363  
Liouville, Joseph, 315, 390; and the work of  
  Galois, 315  
Liu Hui, 86, 89  
Li Ye, 94–95  
Macaulay, Francis, 440  
Maclaurin, Colin, 280–283, 285, 339  
MacMahon, Percy, 372  
Mādhava, 127  
*maestri d'abbaco* (see also *libri d'abbaco*),  
  201–202, 203n68, 205, 223, 228, 229, 237,  
  246  
Mahāvīra, 109; *Ganita-sāra-sangraha* of, 109,  
  113, 118  
Mahoney, Michael, 4–5, 446  
al-Ma'mūn (Caliph), 133, 138  
al-Mansūr (Caliph), 133  
Marcolongo, Roberto, 432  
Maseres, Frances, 400  
Master Dardi of Pisa, 194–197  
*Mathematical Classic of Master Sun*, 101–102  
mathematical induction, 164–165  
matrices, 2, 9, 87–89, 341–342, 347–349,  
  354–355, 427–428; Cayley's work on,  
  356–360, 428; Frobenius's work on,  
  414–415, 428; polynomial algebra of,  
  358–360; Sylvester's coining of the term,  
  356  
Maxwell, James Clerk, 410  
Mazzinghi, Antonio de', 197–200, 206  
Menaechmus, 33  
Mesopotamia, algebra in (see algebra, in  
  Mesopotamia)  
Middle Ages, 174–178  
Möbius, August Ferdinand, 410–411  
modern algebra, 2, 382, 398, 427–447  
module, in the work of Dedekind, 397,  
  432–433  
modulus, 385  
Molien, Theodor, 424–425  
*Moscow Mathematical Papyrus*, 13–16, 83  
Muhammad, 132–133  
Müller, Johannes of Königsberg (also known  
  as Regiomontanus), 227–228  
multilinear forms, 367  
National Mathematics Advisory Panel, 1–2,  
  4–5  
negative numbers, 87–88, 225–227, 400–401  
negative roots of algebraic equations,  
  224–225, 252, 260–261, 268–270  
Netto, Eugen, 326, 331  
Newton, Isaac, 4, 275, 283, 287–288, 313n11,  
  336, 343n12, 399; *Arithmetica universalis*,  
  275–282; quadratic equations in the work  
  of, 278–279  
Nilakanṭha, 127

- nilpotent, 418–420  
*Nine Chapters on the Mathematical Art*, 81, 84–92, 94  
Noether, Emmy, 10, 380, 438–440, 443–444; work of on algebras, 442–443; work of on commutative algebra, 443; work of on ring theory, 434, 437, 440–442  
Noether, Max, 438  
Noetherian rings, 442n25  
nonions, 421  
notational conventions (*see* symbolism in algebra)  
*Nullstellensatz*, 423  
number, expanding concept of in the nineteenth century, 381–399  
Nuñez, Pedro, 205, 235
- octonions, 407–408  
orthogonal transformation, 347, 349, 353, 359, 361
- Pacioli, Luca, 200–201, 214–216, 235, 238; *Summa de arithmetica, geometria, proportioni, e proportionalita*, 200–204, 214–216, 229–230, 233  
*p*-adic numbers, 433, 435  
Pappus of Alexandria, 77; *Mathematical Collection*, 77–78, 237, 253, 266–267  
parabola, Apollonian definition of, 51–52, 247, 257–258, 279–280  
Pascal, Blaise, 162  
Pascal triangle, 92–93, 162–164, 259  
Pazzi, Antonio Maria, 228  
Peacock, George, 401–402, 404  
Peano, Giuseppe, 430; axiomatization of vector spaces of, 430–432  
Peirce, Benjamin, 415–420, 421n100; definition of mathematics of, 416–417  
Peirce, Charles Sanders, 420, 421n100  
Peirce decomposition, 418–419  
Peletier, Jacques, 212, 245, 248  
Pell, John, 122  
Pell equation, 107, 122–126, 130  
Percy, Sir Henry, 249  
perfect numbers, 381  
permutations, 295, 299–300; groups of, 302, 312, 316–318, 321–326, 329–330, 332–333; in the work of Cauchy, 300–303; in the work of Galois, 312–314; in the work of Jordan, 316–317; in the work of Lagrange, 296–299
- Pincherle, Salvatore, 432  
Plato of Tivoli, 178n6  
Poincaré, Henri, 423–424  
polynomial algebra, 2, 145–146, 158–165; of matrices, 358–360  
polynomial equations: in Chinese mathematics, 82, 92–98; of degree five and higher (*see also* quintic equations), 289, 295, 298–300, 303–310, 313–317; method of elimination relative to, 95–98, 209–212, 276, 343, 350–353; in the work of Descartes, 270–271; in the work of Euler, 291–292; in the work of Harriot, 250–252; in the work of Hudde, 272; in the work of Newton, 275; in the work of Viète, 244–246; in the work of Weber, 332  
poristics, 238  
principal axis problem, 346–349, 357–358, 361n49  
Proclus, 33  
proof, notion of, 33–34  
proportions, 7, 12–16, 18–20, 82–86, 107–108  
Pṛthūdakasvāmin, 113, 119–120  
Ptolemy, Claudius, 59, 78, 134, 177  
Ptolemy I, 58  
Pythagoras, 33  
Pythagorean theorem, 38–39, 41–42, 91–92, 97, 110, 152
- Qin Jiushao, 92, 94, 98, 102; *Mathematical Treatise in Nine Sections*, 92–94, 102–104  
quadratic equations, 2, 7–9; in Chinese mathematics, 91–92, 94; in Egyptian mathematics, 15–16; in Indian mathematics, 109–118, 130; in Islamic mathematics, 137–153; in Mesopotamian mathematics, 18, 20–32, 143, 147; six-fold classification of, 139–140, 144; in the work of the cossists, 208, 211–212; in the work of Descartes, 265; in the work of Diophantus, 64–68, 72–77; in the work of Euclid, 35–48; in the work of Girard, 260; in the work of Harriot, 251–252; in the work of Jordanus of Nemore, 188–190; in the work of Leonardo of Pisa, 179–182, 184–187; in the work of the maestri d’abbaco, 192, 197–200; in the work of Newton, 278–279; in the work of Viète, 241, 244–245  
quadratic forms, 318–321, 329, 332, 346–355, 357–361, 364, 373, 385; discriminant of, 318, 353

- quadratic formula, 22, 24–25, 90, 94, 112–114, 130, 142, 149, 180, 190, 241, 251, 265, 276, 279
- quadratic reciprocity, law of, 385–386
- quadric surfaces, 347
- quantics, 370–380
- quartic equations, 246, 283, 297–298; in Chinese mathematics, 93–95, 98; in Indian mathematics, 115–116; in Islamic mathematics, 149–150; in the work of Cardano, 222–225, 227; in the work of Descartes, 268–270; in the work of Diophantus, 74–77; in the work of Girard, 260; in the work of Harriot, 251–252; in the work of the maestri d’abbaco, 196–197; in the work of Pacioli, 214–215; in the work of Viète, 245
- quasi-real-world problems, 147, 151–152, 159–160, 192, 195, 208, 277
- quaternions, 406–407, 415, 420–421, 428; complex, 408
- quintic equations, 289, 314–315; in the work of Bezout, 293–294; in the work of de Moivre, 290–291; in the work of Hermite, 315; unsolvability of, 298–300
- quotient group, definition of by Hölder, 330
- Quṣṭā ibn Lūqā, 61, 80, 156
- Ramée, Pierre de (also known as Peter Ramus), 212–213, 236–237
- Ramus, Peter (*see* Ramée, Pierre de)
- al-Rashīd, Harūn, 133
- real fields, 444
- reciprocity laws (*see* biquadratic reciprocity, law of; cubic reciprocity, law of; higher reciprocity, laws of; quadratic reciprocity, law of)
- Recorde, Robert, 205–206
- Regiomontanus (*see* Müller, Johannes of Königsberg)
- representation theory, 442
- resolvent, 223, 292–294, 296–299
- Rhind Mathematical Papyrus*, 13–15, 32, 77
- Riemann surfaces, 399
- rings, 2, 432, 437; axiomatization of, 433–434, 440–441; work on by Noether, 434, 437, 440–442
- Robert of Chester, 177
- Roman Empire, 59–60, 174–175, 184
- Rompiani, Antonio, 200
- Rudolff, Christoff, 205–209, 235
- Ruffini, Paolo, 298–300, 303
- rule of three, 14; in China, 84–85; in India, 107–108
- St. Petersburg Academy, 291
- Salmon, George, 371, 373, 380
- al-Samaw’al ben Yahyā al-Maghribī, 158, 160–164, 173
- Savasorda (*see* Abraham bar Hiyya of Barcelona)
- Scheffers, Georg, 424–425
- Scheybl, Johann, 221, 235
- Schmidt, Otto, 334
- Schooten, Frans van, 272
- Schreier, Otto, 444
- Schur, Friedrich, 424
- Schur, Issai, 442n26
- Scot, Michael, 185
- series, sums of, 126–130, 164–165
- Serret, Joseph, 315–317, 432
- Shi Huangdi, 81
- Smith, Henry J. S., 363
- solvability by radicals, definition of, 307
- spectral theory, 349, 350n29, 360–366
- square roots, 90–91
- Steinitz, Ernst, 397, 434–437, 443–444
- Stevin, Simon, 233n44
- Stifel, Michael, 209–212, 235, 249
- straight line, equation of, 255–256
- Study, Eduard, 424
- Suan shu shu*, 81–84, 86
- Śulba-sūtras*, 105–106, 113n16; *Baudhāyana-śulba-sūtra*, 109–111
- Sylow, Ludwig, 304n80, 325–326
- Sylvester, James Joseph, 344, 349, 351, 358, 363, 414–415, 416n87, 420–424; coins the term “matrix,” 356; law of inertia of, 349; work of on algebras, 421–424; work of on determinants, 344; work of on invariant theory, 353, 367–373, 375; work of on the theory of forms, 349
- Sylvester II, Pope (*see* Gerbert of Aurillac)
- symbolism in algebra, 2, 4–5, 8–9, 61–62, 69, 173, 204–213, 220n10, 221n12, 230–231, 233–234, 238–239, 241, 243, 249–252, 255–257, 259, 264, 280, 287
- symmetric functions, 259–260, 343–344
- synthesis (*see* analysis and synthesis)
- syzygies, 368–371
- Tait, Peter Guthrie, 410
- Tartaglia, Niccolò (*see* Fontana, Niccolò)
- Thales, 33

- Theatetus, 33  
Theon of Alexandria, 78, 237  
Torporley, Richard, 249  
transformation groups (*see* groups, of transformations)  
transmission of Islamic texts into the Latin West, 177–178  
al-Ṭūsī, Sharaf al-Dīn, 171–173
- unique factorization domain, 383, 386, 387–388
- Vandermonde, Alexandre–Théophile, 294–296, 308n35, 343  
van der Waerden, Bartel, 10, 399, 442–444; *Moderne Algebra*, 10, 444–447  
vectors, 2, 9, 410, 428; Grassmann’s work on, 411–414; Hamilton’s work on, 408–410  
vector spaces, 412–413, 432; axiomatization of by Peano, 430–432  
Viète, François, 8, 62, 236, 245, 245n74, 247–250, 252–253, 256, 258–259, 262, 264, 288, 291, 338; *De Aequationum recognitione et emendatione tractatus duo*, 241–245; definition of algebra of, 236–238; *In artem analyticem isagoge*, 236–240; on Islamic algebra, 237; polynomial equations in the work of, 244–246; work of on quadratic equations, 241, 244–245; *Zeteticorum libri quinque*, 240–241  
von Dyck, Walther, 328–329
- Wallis, John, 249  
Wantzel, Pierre, 309  
Waring, Edward, 294  
Weber, Heinrich, 399, 427, 434–435, 446; polynomial equations in the work of, 332; work of on group theory, 329–333  
Wedderburn, Joseph Henry Maclagan, 419, 426, 437  
Wedderburn-Artin theorem, 426n112  
Weierstrass, Karl, 320, 330, 349, 360, 364, 428, 438n19; work of on algebras, 422–423; work of on the theory of forms, 361–363  
Weil, André, 399  
Weyl Hermann, 432  
Whiston, William, 275, 280  
Wiener, Norbert, 432  
William de Lunis, 178n6  
Witt, Jan de, 272–274
- Yang Hui, 94–95, 99–100, 102
- Zariski, Oscar, 399  
zetetics, 237–238  
Zhang Qujian, 92, 98  
Zhu Shijie, 94–98, 276n52, 344n11  
Zolotarev, Egor, 398n45